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On some differential invariants for a family of diffusion equations

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Abstract
The equivalence transformation algebra L_E and some of its differential invariants for the class of equations $u_t = (h(u)u_x)_x + f(x, u, u_x)$ $(h \neq 0)$ are obtained. Using these invariants, we characterize subclasses which can be mapped by means of an equivalence transformation into the well-studied family of equations $v_t = (v^k v_x)_x$.

1. Introduction

Nonlinear diffusion equations are employed as mathematical models for several phenomena: transport in porous media, thermal conduction, evolution of bacterial colonies, plasma physics, soil water motion, combustion, to cite few.

Several cases of diffusion equations have been deeply studied in the framework of transformation groups (classical and nonclassical symmetries, Bäcklund transformations, approximate symmetries) and a lot of results have been found concerned with the features of some exact solutions or concerned with the integrability of someone of them (see e.g. [1–3]).

In this paper, we deal with the following family of diffusion equations:

$$u_t = (h(u)u_x)_x + f(x, u, u_x), \qquad h \neq 0 \tag{1}$$

in order to find equivalence transformations and some of their differential invariants.

As well known an equivalence transformation for the family under consideration is a non-degenerate change of dependent and independent variables mapping (1) to another equation of the same family but with, in general, different functions f and h. Thus solutions of an equation can be transformed in solutions of an equivalent equation.

Recently the following family

$$u_t = h(x, u)u_{xx} + f(x, u, u_x), \qquad h \neq 0. \tag{2}$$

has been considered by Ibragimov and Sophocleous [4] in order to find differential invariants.

As an application of differential invariants, the nonlinear equation

$$u_t = u_{xx} + f(u, u_x) \tag{3}$$

has been studied in [5] in order to find classes of linearizable equations.

We show that the knowledge of the differential invariants is useful in order to find a subclass of equations (1) which can be brought by an equivalence transformation into a well-studied specific target equation.

The plan of the paper is the following. In the following section we obtain the equivalence algebra for family (1). In section 3 we look for differential invariants with respect to the equivalence group G_E. In section 4 we use these latter ones to obtain subclasses of equations (1) which can be mapped in the well-known family

$$v_t = (v^k v_x)_x. \tag{4}$$

In section 5 we apply the obtained results to show as is possible to get solutions from the solutions of the target equation. The conclusions are shown in section 6.

2. Equivalence algebra

As is known an equivalence transformation for the class under consideration is an invertible transformation of the independent and dependent variables,

$$t = \alpha(\hat{t}, \hat{x}, v), \qquad x = \beta(\hat{t}, \hat{x}, v), \qquad u = \gamma(\hat{t}, \hat{x}, v), \tag{5}$$

that changes equations (1) into equations of the same form

$$v_{\hat{t}} = (\hat{h}(v)v_{\hat{x}})_{\hat{x}} + \hat{f}(\hat{x}, v, v_{\hat{x}}), \tag{6}$$

where in general $(\hat{h}, \hat{f}) \neq (h, f)$. Our goal is to find an equivalence algebra for the family of equations (1). In order to obtain continuous groups of equivalence transformations of equations (1) we consider the arbitrary functions f and h as dependent variables and apply, as suggested by Ovsiannikov [6], the Lie infinitesimal invariance criterion to the following system:

$$\begin{aligned} &(h(u)u_x)_x + f(x, u, u_x) - u_t = 0, \\ &f_t = f_{u_t} = 0, \\ &h_t = h_x = h_{u_t} = h_{u_x} = 0. \end{aligned} \tag{7}$$

Then we search for the equivalence operator Y in the following form:

$$Y = \xi^1\partial_t + \xi^2\partial_x + \eta\partial_u + \zeta_1\partial_{u_t} + \zeta_2\partial_{u_x} + \mu_1\partial_f + \mu_2\partial_h, \tag{8}$$

which applied to equations (7) leaves them invariants.

In (8) ξ^1, ξ^2 and η are sought depending on t, x and u, while μ_1 and μ_2 depend on t, x, u, u_t, u_x, f and h, and the components ζ_1 and ζ_2, as is known, are given by

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \zeta_2 &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2). \end{aligned} \tag{9}$$

The operators D_t and D_x denote the total derivatives with respect to t and x:

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots, \tag{10}$$

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots. \tag{11}$$

The prolongation of operator (8), which we need is

$$\widetilde{Y} = Y + \zeta_{22}\partial_{u_{xx}} + \omega^1_t\partial_{f_t} + \omega^1_{u_t}\partial_{f_{u_t}} + \omega^2_u\partial_{h_u} + \omega^2_t\partial_{h_t} + \omega^2_x\partial_{h_x} + \omega^2_{u_t}\partial_{h_{u_t}} + \omega^2_{u_x}\partial_{h_{u_x}}, \tag{12}$$

where (see e.g. [7, 8])

$$\zeta_{22} = D_x(\zeta_2) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), \tag{13}$$

$$\omega_t^1 = \widetilde{D}_t(\mu_1) - f_x \widetilde{D}_t(\xi^2) - f_u \widetilde{D}_t(\eta) - f_{u_x} \widetilde{D}_t(\zeta_1), \tag{14}$$

$$\omega_{u_t}^1 = \widetilde{D}_{u_t}(\mu_1) - f_x \widetilde{D}_{u_t}(\xi^2) - f_u \widetilde{D}_{u_t}(\eta) - f_{u_x} \widetilde{D}_{u_t}(\zeta_1), \tag{15}$$

$$\omega_t^2 = \widetilde{D}_t(\mu_2) - h_u \widetilde{D}_t(\eta), \tag{16}$$

$$\omega_x^2 = \widetilde{D}_x(\mu_2) - h_u \widetilde{D}_x(\eta), \tag{17}$$

$$\omega_{u_t}^2 = \widetilde{D}_{u_t}(\mu_2) - h_u \widetilde{D}_{u_t}(\eta), \tag{18}$$

$$\omega_{u_x}^2 = \widetilde{D}_{u_x}(\mu_2) - h_u \widetilde{D}_{u_x}(\eta), \tag{19}$$

while $\widetilde{D}_t$, $\widetilde{D}_x$, $\widetilde{D}_u$, $\widetilde{D}_{u_t}$ and $\widetilde{D}_{u_x}$ are defined by

$$\widetilde{D}_t = \partial_t, \tag{20}$$

$$\widetilde{D}_x = \partial_x + f_x \partial_f + f_{xx} \partial_{f_x} + \cdots, \tag{21}$$

$$\widetilde{D}_u = \partial_u + f_u \partial_f + h_u \partial_h + f_{ux} \partial_{f_x} + h_{uu} \partial_{h_u} + f_{uu} \partial_{f_u} + \cdots, \tag{22}$$

$$\widetilde{D}_{u_t} = \partial_{u_t}, \tag{23}$$

$$\widetilde{D}_{u_x} = \partial_{u_x} + f_{u_x} \partial_f + f_{xu_x} \partial_{f_x} + f_{uu_x} \partial_{f_u} + \cdots. \tag{24}$$

After having applied operator (12) to system (7) and following the well-known algorithm (see e.g. [7, 9–11]), we found that the class of equations (1) admits an infinite continuous group G_E of equivalence transformations generated by the Lie algebra L_E spanned by the operators

$$Y_0 = t\partial_t - f\partial_f - h\partial_h - u_t \partial_{u_t}, \tag{25}$$

$$Y_1 = \partial_t, \tag{26}$$

$$Y_2 = \partial_x, \tag{27}$$

$$Y_3 = x\partial_x + 2h\partial_h - u_x \partial_{u_x}, \tag{28}$$

$$Y_\varphi = \varphi\partial_u + \left(\varphi_u f - h u_x^2 \varphi_{uu}\right) \partial_f + \varphi_u u_t \partial_{u_t} + \varphi_u u_x \partial_{u_x}, \tag{29}$$

where φ is an arbitrary function of u.

3. Search for differential invariants

A differential invariant of order s for family (1) is a real-valued function J of the independent variables t, x, the dependent variable u and its derivatives u_t, u_x, as well as of the functions f, h and their derivatives of maximal order s, that is invariant with respect to the equivalence group G_E.

That is, by using the infinitesimal method [8, 12], J is a non-constant function which satisfy the PDE system

$$Y_i^{(s)}(J) = 0 \qquad i = 0, 1, 2, 3, \varphi \tag{30}$$

where $Y_i^{(s)}$ are the sth prolongation of Y_i.

We seek for differential invariants of zero order, i.e. non-constant functions of the form

$$J = J(t, x, u, u_t, u_x, f, h), \tag{31}$$

which are invariant with respect to the equivalence group G_E.

By applying the invariant test $Y(J) = 0$ we do not find differential invariants of zero order, but the invariant equations $h = 0$, $u_t = 0$ and $u_x = 0$.

In order to look for differential invariants of first order

$$J = J(t, x, u, u_t, u_x, f, h, f_x, f_u, f_{u_x}, h_u), \tag{32}$$

we need the following first prolongation of operator Y:

$$Y^{(1)} = Y + \omega_x^1 \partial_{f_x} + \omega_u^1 \partial_{f_u} + \omega_{u_x}^1 \partial_{f_{u_x}} + \omega_u^2 \partial_{h_u}, \tag{33}$$

where $\omega_x^1, \omega_u^1, \omega_{u_x}^1$ and ω_u^2 likewise (14) . . . (19).

Applying the invariant test $Y^{(1)}(J) = 0$, after some calculation, we get

$$J = J(\lambda_1, \lambda_2, \lambda_3), \tag{34}$$

where

$$\lambda_1 = \frac{u_x}{u_t^2} h f_x, \tag{35}$$

$$\lambda_2 = \frac{2f - u_x f_{u_x}}{2u_t}, \tag{36}$$

$$\lambda_3 = \frac{u_x^2 h_u}{u_t}. \tag{37}$$

4. Some classes of equivalent equations

Here, we consider the family of equations

$$v_t = (v^k v_x)_x \qquad k \in \mathbf{R} \tag{38}$$

belonging to (1). This family has largely been studied in the framework of heat conduction and porous material [13, 14]. Wide classes of solutions have been found (see e.g. [15, 16]) and for $k = -2$ it has been linearized [17].

Now, we look for equations of class (1) which are equivalent to (38).

We recall that two equivalent equations of class (1) have the same differential invariants with respect to G_E.

For the target equations (38) we get

$$\lambda_1 = \lambda_2 = 0, \tag{39}$$

while

$$\lambda_3 = \frac{u_x^2 k u^{k-1}}{u_t}. \tag{40}$$

So, from (40), two cases arise:

(i) $k = 0$ and $\lambda_3 = 0$;
(ii) $k \neq 0$ and $\lambda_3 \neq 0$.

We analyse them separately.

4.1. $k = 0$

In this case our target becomes

$$v_t = v_{xx}. \tag{41}$$

So the arbitrary elements f and h of class (1) must satisfy the following conditions:

$$\begin{cases} \lambda_1 \equiv \dfrac{u_x}{u_t^2} h f_x = 0 \\ \lambda_2 \equiv \dfrac{2f - u_x f_{u_x}}{2u_t} = 0 \\ \lambda_3 \equiv \dfrac{u_x^2 h_u}{u_t} = 0. \end{cases} \tag{42}$$

Then the more general form of equations (1), having the same invariant of the target, is the generalized potential Burger's equations

$$u_t = h_0 u_{xx} + u_x^2 g(u), \tag{43}$$

where $h_0 \neq 0$ is an arbitrary constant and g is an arbitrary function of u.

Equations (43) are a subclass of the family

$$u_t = h_0 u_{xx} + g(u)|u_x|^{p-1} u_x, \tag{44}$$

widely considered in [18, 19].

Since conditions (42) are invariant with respect to the equivalence group, all equations of class (43) are transformed by an equivalence transformation into equations of the same form.

So it is possible to find at least an equivalence transformation which maps an equation of form (43) into equation (41).

To this aim following [5] we use, for instance, an equivalence transformation of the form

$$t = h_0 \hat{t}, \qquad x = h_0 \hat{x}, \qquad u = \psi(v(\hat{t}, \hat{x})).$$

By applying this transformation to equation (43) and requiring that the transformed equation must be of form (41), we get that ψ must be solution of the following differential equation:

$$\frac{\psi''}{\psi'} h_0 + \psi' g(\psi) = 0. \tag{45}$$

Now, we are able to affirm the following.

Theorem 1. *An equation belonging to class (1) can be transformed by an equivalence transformation of the group G_E into the linear equation*

$$v_t = v_{xx}, \tag{46}$$

if and only if the function h is a constant and f is given by

$$f = u_x^2 g(u). \tag{47}$$

4.2. $k \neq 0$

In this case, from

$$\begin{cases} \lambda_1 \equiv \dfrac{u_x}{u_t^2} h f_x = 0 \\ \lambda_2 \equiv \dfrac{2f - u_x f_{u_x}}{2u_t} = 0, \end{cases} \tag{48}$$

we get the functional form of $f(x, u, u_x)$ which reads

$$f = u_x^2 g(u), \tag{49}$$

with g being an arbitrary function of u.

It is a simple matter to see that any equation of class (1) having the form

$$u_t = (h(u)u_x)_x + u_x^2 g(u) \tag{50}$$

can be mapped by an equivalence transformation into an equation of the same form.

The equation of form (50) falls in the following known family:

$$u_t = \nabla \cdot (\alpha(u)\nabla u) + \beta(u)\nabla u^2 \tag{51}$$

considered in [20], in the one-dimensional case.

In order to look for the equations of form (50) which can be brought by an equivalence transformation in the form of the well-known studied subclass (38), we search for differential invariants for the family of equations (50).

This search implies that we must look for functions of the form

$$J = J(t, x, u, u_t, u_x, h, h_u, h_{uu}, \ldots, g, g_u, g_{uu}, \ldots), \tag{52}$$

which are invariant with respect to the equivalence generator of (50) (which reads)

$$\Upsilon = \bar{\xi}^1\partial_t + \bar{\xi}^2\partial_x + \bar{\eta}\partial_u + \bar{\zeta}_1\partial_{u_t} + \bar{\zeta}_2\partial_{u_x} + \bar{\mu}_1\partial_h + \bar{\mu}_2\partial_g. \tag{53}$$

In order to write the infinitesimal components of Υ we use the algorithm proposed in [21, 22]. In fact, by knowing the infinitesimal components of the equivalence generator Y, we are able to find the corresponding new coordinates $\bar{\xi}^1, \bar{\xi}^2, \bar{\eta}, \bar{\mu}_1, \bar{\mu}_2$ by making the changes of coordinates

$$\bar{t} = t, \tag{54}$$

$$\bar{x} = x, \tag{55}$$

$$\bar{u} = u, \tag{56}$$

$$\bar{h} = h, \tag{57}$$

$$\bar{u}_{\bar{x}}^2 g(\bar{u}) = f \tag{58}$$

and by requiring their invariance with respect to

$$\begin{aligned} Y^* &\equiv Y + \Upsilon \\ &\equiv \xi^1\partial_t + \xi^2\partial_x + \eta\partial_u + \zeta^1\partial_{u_t} + \zeta^2\partial_{u_x} + \mu_1\partial_h + \mu_2\partial_f \\ &\quad + \bar{\xi}^1\partial_t + \bar{\xi}^2\partial_x + \bar{\eta}\partial_u + \bar{\zeta}^1\partial_{u_t} + \bar{\zeta}^2\partial_{u_x} + +\bar{\mu}_1\partial_h + \bar{\mu}_2\partial_g. \end{aligned} \tag{59}$$

In this way, we get

$$\bar{\xi}^1 = \xi^1, \tag{60}$$

$$\bar{\xi}^2 = \xi^2, \tag{61}$$

$$\bar{\eta} = \eta, \tag{62}$$

$$\bar{\zeta}^1 = \zeta^1 \tag{63}$$

$$\bar{\zeta}^2 = \zeta^2 \tag{64}$$

$$\bar{\mu}_1 = \mu_1 \tag{65}$$

$$\bar{\mu}_2 = g(2c_3 - c_0 - \phi_u - h\phi_{uu}). \tag{66}$$

Then the new equivalence generator for class (50) reads

$$\begin{aligned} \Upsilon = {} & (c_1 + c_0 t)\partial_t + (c_3 x + c_2)\partial_x + \varphi(u)\partial_u + u_t(\varphi_u - c_0)\partial_{u_t} + (\varphi_u - c_3)u_x\partial_{u_x} \\ & + [g(2c_3 - c_0 - \varphi_u) - h\varphi_{uu}]\partial_g + h(2c_3 - c_0)\partial_h. \end{aligned} \tag{67}$$

So, after performing the invariant test, by using the prolongations of the operator Υ, we get one first-order differential invariant

$$\alpha = \frac{u_x^2 h_u}{u_t} \equiv \lambda_3 \tag{68}$$

and one second-order differential invariant

$$\beta = \frac{u_x^4}{u_t^2}\left(g h_u - h h_{uu}\right). \tag{69}$$

We observe that equation (38) satisfies the following invariant equation:

$$(k-1)\alpha^2 + k\beta \equiv (k-1)\frac{u_x^4 h_u^2}{u_t^2} + k\frac{u_x^4(g h_u - h h_{uu})}{u_t^2} = 0. \tag{70}$$

An equation belonging to class (50) can be mapped into an equation of form (38) if the functions h and g satisfy condition (70), i.e. if

$$g(u) = \frac{h h_{uu}}{h_u} - \frac{k-1}{k} h_u, \tag{71}$$

we can say that there exists at least an equivalence transformation mapping the equations

$$u_t = (h(u) u_x)_x + u_x^2\left(\frac{h h_{uu}}{h_u} - \frac{k-1}{k} h_u\right) \tag{72}$$

into (38).

In fact if we consider, for instance, the invertible transformation

$$u = \psi(v) \tag{73}$$

by applying it to equations (72), we get

$$v_t = (h(\psi) v_x)_x + v_x^2\left[h\frac{\psi''}{\psi'} + \psi'\frac{h h_{\psi\psi}}{h_\psi} - \frac{k-1}{k}\psi' h_\psi\right]. \tag{74}$$

By choosing

$$\psi = h^{-1}(v^k), \tag{75}$$

we obtain equation (38).

So we showed that it exists at least one invertible transformation between (38) and (72). We are able, now, to affirm the following.

Theorem 2. *An equation belonging to class (1) can be mapped by an equivalence transformation of the group G_E into the well-known equation*

$$v_t = (v^k v_x)_x \tag{76}$$

if and only if the function f is given by

$$f = u_x^2\left(\frac{h h_{uu}}{h_u} - \frac{k-1}{k} h_u\right). \tag{77}$$

5. Applications

In this section we wish to apply the result obtained in section 4.2.

As is well known an equivalence transformation maps solutions into solutions of transformed equation [10].

Consequently, the solutions of (38) can be mapped to the solutions of (72) and vice versa.

In the following, we show some subclasses which could be brought to the well-studied nonlinear diffusion equation (38).

(i) If we put $k = -2$, (71) becomes

$$g(u) = \frac{hh_{uu}}{h_u} - \frac{3}{2}h_u, \tag{78}$$

then all the equations of form (50) with $h(u)$ and $g(u)$ satisfying the above condition, following [17], are linearizable.

(ii) If we put $k = 3$, (71) becomes

$$g(u) = \frac{hh_{uu}}{h_u} - \frac{2}{3}h_u. \tag{79}$$

So it is possible to bring all equations, whose $h(u)$ and $g(u)$ satisfy (79), to the form

$$v_t = (v^3 v_x)_x, \tag{80}$$

well studied as the thin films spreading under gravity equation [23].

(iii) If we put $k = 1$, (71) becomes

$$g(u) = \frac{hh_{uu}}{h_u}. \tag{81}$$

In this case the equation characterized by (81) can be put in the form

$$v_t = (v v_x)_x \tag{82}$$

studied as the thin saturated regions in the porous media equation [24].

(iv) If we put $k = 6$, (71) becomes

$$g(u) = \frac{hh_{uu}}{h_u} - \frac{5}{6}h_u. \tag{83}$$

In this case the equation characterized by (83) can be put in the form

$$v_t = (v^6 v_x)_x, \tag{84}$$

known as the equation for radiative heat transfer by the Marshak waves equation [25].

If we consider the well-known invariant solutions of (38) [15], by denoting them with $v(t, x)$ and taking into account that $u(t, x) = h^{-1}(v^k)$, we get the following.

(i) Stationary solutions

$$v = (a_1 x + a_2)^{\frac{1}{k+1}}, \qquad u = h^{-1}\left((a_1 x + a_2)^{\frac{k}{k+1}}\right), \qquad k \neq -1, \tag{85}$$

$$v = a_2\,\mathrm{e}^{a_1 x} \qquad u = h^{-1}((a_2\,\mathrm{e}^{a_1 x})^{-1}), \qquad k = -1. \tag{86}$$

(ii) Travelling wave solutions

$$v = [k\lambda(\lambda t - x) + a_1]^{1/k}, \qquad u = h^{-1}(k\lambda(\lambda t - x) + a_1). \tag{87}$$

(iii) A self-similar solution

$$v = \left\{a_1|t - T_0|^{\frac{-k}{k+2}} + \frac{k}{2(k+2)}\frac{x^2}{T_0 - t}\right\}^{1/k}, \tag{88}$$

$$u = h^{-1}\left(a_1|t - T_0|^{\frac{-k}{k+2}} + \frac{k}{2(k+2)}\frac{x^2}{T_0 - t}\right). \tag{89}$$

It is worth stressing that we took into consideration the invariant solutions only for the simplicity sake.

As an example we consider the non-stationary seepage equation in the one-dimensional case [26],

$$u_t = \frac{\alpha\gamma}{m} u^{\frac{\gamma}{\gamma+1}} u_{xx}, \tag{90}$$

where m is the porosity of the medium, α is the seepage coefficient, γ is the polytropic exponent and u is the pressure in the medium.

Equation (90) falls in class (1) with $h(u) = \frac{\alpha\gamma}{m} u^{\frac{\gamma}{\gamma+1}}$ and $f(x, u, u_x) = -\frac{\alpha\gamma^2}{m(\gamma+1)} u^{-\frac{1}{\gamma+1}} u_x^2$. In order to find the target equation of the form

$$v_t = (v^k v_x)_x, \tag{91}$$

we consider relation (71) with $g(u) = -\frac{\alpha\gamma^2}{m(\gamma+1)} u^{-\frac{1}{\gamma+1}}$, from where we get

$$-\frac{\alpha\gamma^2}{m(\gamma+1)} u^{-\frac{1}{\gamma+1}} = -\alpha\gamma \frac{k\gamma - \gamma + k}{mk(\gamma+1)} u^{-\frac{1}{\gamma+1}}, \tag{92}$$

which implies $k = \gamma$.

After observing that $h^{-1}(v^\gamma) = \left(\frac{m}{\alpha\gamma}\right)^{\frac{\gamma+1}{\gamma}} v^{\gamma+1}$, in our case, by using some of the invariant solutions of

$$v_t = (v^\gamma v_x)_x, \tag{93}$$

we get the following special class of the solutions of equation (90).

(i) Stationary solutions

$$u = \left(\frac{m}{\alpha\gamma}\right)^{\frac{\gamma+1}{\gamma}} (a_1 x + a_2). \tag{94}$$

(ii) Travelling wave solutions

$$u = \left(\frac{m}{\alpha\gamma}\right)^{\frac{\gamma+1}{\gamma}} [\gamma\lambda(\lambda t - x) + a_1]^{\frac{\gamma+1}{\gamma}}. \tag{95}$$

(iii) A self-similar solution

$$u = \left(\frac{m}{\alpha\gamma}\right)^{\frac{\gamma+1}{\gamma}} \left\{ a_1 |t - T_0|^{\frac{-\gamma}{\gamma+2}} + \frac{\gamma}{2(\gamma+2)} \frac{x^2}{T_0 - t} \right\}^{\frac{\gamma+1}{\gamma}}. \tag{96}$$

6. Conclusions

After having found the equivalence transformations for the class of equations under consideration, we look for some differential invariants of equivalence transformations by restricting ourselves to those of the first order. We use these latter to characterize the form of the equations of class (1) equivalent to the equation

$$u_t = (u^k u_x)_x. \tag{97}$$

For $k = 0$, equation (97) becomes the linear Fourier's equation, and we showed that it is possible to find at least an equivalence transformation which maps the following generalized Burger's potential equations:

$$u_t = h_0 u_{xx} + u_x^2 g(u) \tag{98}$$

into the linear Fourier's equation.

For $k \neq 0$, we showed that equations (97) are equivalent to the following subclasses of equations:

$$u_t = (h(u)u_x)_x + u_x^2\left(\frac{hh_{uu}}{h_u} - \frac{k-1}{k}h_u\right). \tag{99}$$

In this case also we are able to obtain at least an equivalence transformation which bring (99) into (97).

Finally, we applied this transformation in order to bring the known solutions of equations (97) into the solutions of equations (99).

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